

LIPSCHITZ TYPE SPACES AND LANDAU-BLOCH TYPE THEOREMS FOR HARMONIC FUNCTIONS AND POISSON EQUATIONS

SH. CHEN, M. MATELJEVIĆ, S. PONNUSAMY[†], AND X. WANG

ABSTRACT. In this paper, we investigate some properties on harmonic functions and solutions to Poisson equations. First, we will discuss the Lipschitz type spaces on harmonic functions. Secondly, we establish the Schwarz-Pick lemma for harmonic functions in the unit ball \mathbb{B}^n of \mathbb{R}^n , and then we apply it to obtain a Bloch theorem for harmonic functions in Hardy spaces. At last, we use a normal family argument to extend the Landau-Bloch type theorem to functions which are solutions to Poisson equations.

1. INTRODUCTION AND MAIN RESULTS

Let \mathbb{R}^n denote the usual real vector space of dimension n , where $n \geq 2$ is a positive integer. Sometimes it is convenient to identify each point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ with an $n \times 1$ column matrix so that

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

For $a = (a_1, \dots, a_n)$ and $x \in \mathbb{R}^n$, we define the Euclidean inner product $\langle \cdot, \cdot \rangle$ by

$$\langle x, a \rangle = x_1 a_1 + \dots + x_n a_n$$

so that the Euclidean length of x is defined by

$$|x| = \langle x, x \rangle^{1/2} = (|x_1|^2 + \dots + |x_n|^2)^{1/2}.$$

Denote a ball in \mathbb{R}^n with center x' and radius r by

$$\mathbb{B}^n(x', r) = \{x \in \mathbb{R}^n : |x - x'| < r\}.$$

In particular, \mathbb{B}^n denotes the unit ball $\mathbb{B}^n(0, 1)$. Set $\mathbb{D} = \mathbb{B}^2$, the open unit disk in the complex plane \mathbb{C} .

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[†] This author is on leave from the Department of Mathematics, Indian Institute of Technology Madras, Chennai-600 036, India.

A function f of an open subset $\Omega \subset \mathbb{R}^n$ into \mathbb{R} is called a *harmonic function* if $\Delta f = 0$, where Δ represents the n -dimensional Laplacian operator

$$\Delta = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}.$$

In this paper, we use C to denote the various positive constants, whose value may change from one occurrence to the next.

A continuous increasing function $\omega : [0, \infty) \rightarrow [0, \infty)$ with $\omega(0) = 0$ is called a *majorant* if $\omega(t)/t$ is non-increasing for $t > 0$. Given a subset Ω of \mathbb{R}^n , a function $f : \Omega \rightarrow \mathbb{R}^m$ ($m \geq 1$) is said to belong to the *Lipschitz space* $\Lambda_\omega(\Omega)$ if there is a positive constant C such that

$$(1) \quad |f(x) - f(y)| \leq C\omega(|x - y|) \quad \text{for all } x, y \in \Omega.$$

For $\delta_0 > 0$, let

$$(2) \quad \int_0^\delta \frac{\omega(t)}{t} dt \leq C \cdot \omega(\delta), \quad 0 < \delta < \delta_0$$

and

$$(3) \quad \delta \int_\delta^\infty \frac{\omega(t)}{t^2} dt \leq C \cdot \omega(\delta), \quad 0 < \delta < \delta_0,$$

where ω is a majorant. A majorant ω is said to be *regular* if it satisfies the conditions (2) and (3) (see [12, 13, 14, 29, 30, 31]).

Let Ω be a domain in \mathbb{R}^n with non-empty boundary. We use $d_\Omega(x)$ to denote the Euclidean distance from x to the boundary $\partial\Omega$ of Ω . In particular, we always use $d(x)$ to denote the Euclidean distance from x to the boundary of \mathbb{B}^n .

A proper subdomain G of \mathbb{R}^n is said to be Λ_ω -*extension* if $\Lambda_\omega(G) = \text{loc}\Lambda_\omega(G)$, where $\text{loc}\Lambda_\omega(G)$ denotes the set of all functions $f : G \rightarrow \mathbb{R}^m$ satisfying (1) with a fixed positive constant C , whenever $x \in G$ and $y \in G$ such that $|x - y| < \frac{1}{2}d_G(x)$. Obviously, \mathbb{B}^n is a Λ_ω -extension domain.

In [21], the author proved that G is a Λ_ω -extension domain if and only if each pair of points $x, y \in G$ can be joined by a rectifiable curve $\gamma \subset G$ satisfying

$$(4) \quad \int_\gamma \frac{\omega(d_G(\zeta))}{d_G(\zeta)} ds(\zeta) \leq C\omega(|x - y|)$$

with some fixed positive constant $C = C(G, \omega)$, where ds stands for the arc length measure on γ . Furthermore, Lappalainen [21, Theorem 4.12] proved that Λ_ω -extension domains exist only for majorants ω satisfying (2). See [14, 16, 19, 21] for more details on Λ_ω -extension domains.

Krantz [20] proved a Hardy-Littlewood type theorem for harmonic functions in the unit ball with respect to the majorant $\omega(t) = \omega_\alpha(t) = t^\alpha$ ($0 < \alpha \leq 1$) as follows.

Theorem A. ([20, Theorem 15.8]) *Let f be a harmonic function from \mathbb{B}^n into \mathbb{R} and $0 < \alpha \leq 1$. Then f satisfies*

$$|\nabla f(x)| \leq C \frac{\omega_\alpha(d(x))}{d(x)} \quad \text{for any } x \in \mathbb{B}^n$$

if and only if

$$|f(x) - f(y)| \leq C\omega_\alpha(|x - y|) \text{ for any } x, y \in \mathbb{B}^n,$$

where ∇f denotes the gradient of f .

For the extensive discussions on this topic, see [1, 2, 3, 4, 8]. We generalize Theorem A to the following form.

Theorem 1. *Let ω be a majorant satisfying (2), Ω be a Λ_ω -extension domain in \mathbb{R}^n and f be a harmonic function from Ω into \mathbb{R} . Then $f \in \Lambda_\omega(\Omega)$ if and only if*

$$|\nabla f(x)| \leq C \frac{\omega(d_\Omega(x))}{d_\Omega(x)} \text{ for any } x \in \Omega.$$

In [18], Holland-Walsh obtained the following result. For the extensive studies on this topic, see [9, 32, 34].

Theorem B. ([18, Theorem 3]) *Let \mathcal{B} denote all analytic functions in \mathbb{D} which form a complex Banach space with the norm*

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{D}} \{(1 - |z|^2)|f'(z)|\} < \infty.$$

Then $f \in \mathcal{B}$ if and only if

$$\sup_{z, w \in \mathbb{D}, z \neq w} \left\{ \frac{\sqrt{(1 - |z|^2)(1 - |w|^2)}|f(z) - f(w)|}{|z - w|} \right\} < \infty.$$

In [32], Pavlović generalized Theorem B into the following form.

Theorem C. ([32, Theorem 2]) *Let $\mathcal{C}^1(\mathbb{B}^n)$ be the class of all one order continuous differentiable functions from \mathbb{B}^n into \mathbb{R} . Let $\mathcal{B}_{\mathcal{C}^1}$ denote all $f \in \mathcal{C}^1(\mathbb{B}^n)$ which form a Banach space with the norm*

$$\|f\|_{\mathcal{B}_{\mathcal{C}^1}} = |f(0)| + \sup_{x \in \mathbb{B}^n} \{(1 - |x|^2)|\nabla f(x)|\} < \infty.$$

Then $f \in \mathcal{B}_{\mathcal{C}^1}$ if and only if

$$\sup_{x, y \in \mathbb{B}^n, x \neq y} \left\{ \frac{\sqrt{(1 - |x|^2)(1 - |y|^2)}|f(x) - f(y)|}{|x - y|} \right\} < \infty.$$

By using a different proof methods, we will prove a more general result as follows which is a generalization of Theorems B and C.

Theorem 2. *Let $f \in \mathcal{C}^1(\mathbb{B}^n)$ and ω be a majorant. Then for any $x \in \mathbb{B}^n$,*

$$|\nabla f(x)| \leq C\omega\left(\frac{1}{d(x)}\right)$$

if and only if for any $x, y \in \mathbb{B}^n$ with $x \neq y$,

$$\frac{|f(x) - f(y)|}{|x - y|} \leq C\omega\left(\frac{1}{\sqrt{d(x)d(y)}}\right).$$

Dyakonov [13] discussed the relationship between the Lipschitz space and the bounded mean oscillation on holomorphic functions in \mathbb{D} , and obtained the following result.

Theorem D. ([13, Theorem 1]) *Suppose that f is a holomorphic function in \mathbb{D} which is continuous up to the boundary of \mathbb{D} . If ω and ω^2 are regular majorants, then*

$$f \in \Lambda_\omega(\mathbb{D}) \iff P_{|f|^2}(z) - |f(z)|^2 \leq M\omega^2(d(z)),$$

where

$$P_{|f|^2}(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|z - e^{i\theta}|^2} |f(e^{i\theta})|^2 d\theta.$$

In particular, for harmonic functions, we get the following result which is analogous to Theorems B and D. For some related topics on complex-valued functions, we refer to [11, 12].

Theorem 3. *Let $f \in C^1(\mathbb{B}^n)$ be a harmonic and ω be a majorant. Then the following are equivalent:*

(a) *for any $x \in \mathbb{B}^n$,*

$$|\nabla f(x)| \leq C\omega\left(\frac{1}{d(x)}\right);$$

(b) *for any $x, y \in \mathbb{B}^n$ with $x \neq y$,*

$$\frac{|f(x) - f(y)|}{|x - y|} \leq C\omega\left(\frac{1}{\sqrt{d(x)d(y)}}\right);$$

(c) *for any $r \in (0, d(x))$,*

$$\frac{1}{|\mathbb{B}^n(x, r)|} \int_{\mathbb{B}^n(x, r)} |f(\zeta) - f(x)| dV(\zeta) \leq Cr\omega\left(\frac{1}{r}\right),$$

where dV denotes the Lebesgue volume measure in \mathbb{B}^n .

For a vector-valued and real harmonic function $f = (f_1, \dots, f_n)$ from \mathbb{B}^n into \mathbb{R}^n (i.e. for each $i \in \{1, 2, \dots, n\}$, $f_i : \mathbb{B}^n \rightarrow \mathbb{R}$ is harmonic), we denote the Jacobian of f by J_f , i.e.,

$$J_f = \det \left(\frac{\partial f_i}{\partial x_j} \right)_{n \times n},$$

where $j \in \{1, 2, \dots, n\}$. Let $\mathcal{H}(\mathbb{B}^n, \mathbb{R}^n)$ be the set of all real harmonic functions f from \mathbb{B}^n into \mathbb{R}^n . Also, for $p \in (0, \infty)$, let $\mathcal{H}^p(\mathbb{B}^n, \mathbb{R}^n)$ denote the harmonic Hardy class consisting of all functions $f \in \mathcal{H}(\mathbb{B}^n, \mathbb{R}^n)$ such that

$$\|f\|_p = \sup_{0 < r < 1} M_p(f, r) < \infty, \quad M_p^p(f, r) := \int_{\partial \mathbb{B}^n} |f(r\zeta)|^p d\sigma(\zeta),$$

where $d\sigma$ is the normalized surface measure on $\partial \mathbb{B}^n$ (see [5]).

One of the long standing open problems in geometric function theory is to determine the precise value of the univalent Landau-Bloch constant for analytic functions of \mathbb{D} . It has attracted much attention, see [23, 26, 27, 28] and references therein.

For general holomorphic mappings of more than one complex variable, no univalent Landau-Bloch constant exists (cf. [37]). In order to obtain some analogous results of univalent Landau-Bloch constant for functions with several complex variables, it is necessary to restrict the class of mappings considered, see [7, 10, 15, 22, 24, 35, 37].

In [6], the authors discussed the Schwarz-Pick Lemma and the Landau-Bloch type theorems for bounded pluriharmonic mappings. It is known that pluriharmonic mappings are special vector-valued harmonic functions. By using a different approach, as our last aim, we will establish the Schwarz-Pick Lemma and obtain a univalent Landau-Bloch constant for vector-valued harmonic functions in the Hardy spaces. Since all bounded vector-valued harmonic functions belong to the harmonic Hardy classes, we see that our result (Theorem 4) is a generalization of [6, Theorem 5].

Theorem 4. *Suppose that $f \in \mathcal{H}^p(\mathbb{B}^n, \mathbb{R}^n)$ satisfies $J_f(0) - 1 = |f(0)| = 0$, where $p \geq 1$ and $n \geq 3$. Then $f(\mathbb{B}^n)$ contains a univalent ball $\mathbb{B}^n(0, R)$, where*

$$R \geq \max_{0 < r < 1} \varphi(r),$$

where

$$\varphi(r) = \frac{1}{2[nK(r)]^{2n-2}M(r)[(1 + \sqrt{2})^{n-1} + \sqrt{2} - 1]},$$

$$K(r) = 2^{1/p}\|f\|_p/[r(1-r)^{(n-1)/p}] \text{ and } M(r) = K(r)[(3 + \sqrt{3})n + 2\sqrt{2}].$$

We remark that, as $\lim_{r \rightarrow 0+} \varphi(r) = \lim_{r \rightarrow 1-} \varphi(r) = 0$, the maximum of $\varphi(r)$ in Theorem 4 does exist.

The following result easily follows from Theorem 4.

Theorem 5. *Let $f \in \mathcal{H}(\mathbb{B}^n, \mathbb{R}^n)$ with $J_f(0) - 1 = |f(0)| = 0$ and $|f(x)| < M$ for $x \in \mathbb{B}^n$. Then f is univalent in $\mathbb{B}^n(0, \rho_0)$ and $f(\mathbb{B}^n(0, \rho_0))$ contains a univalent ball $\mathbb{B}^n(0, R_0)$, where*

$$\rho_0 = \frac{1}{n^{n-1}M^n[(3 + \sqrt{2})n + 2\sqrt{2}][(1 + \sqrt{2})^{n-1} + \sqrt{2} - 1]} \text{ and } R_0 = \frac{\rho_0}{2(nM)^{n-1}}.$$

We will extend Theorem 5 to a general case. Let us give some preparations before we present our next result.

Let $f : \overline{\Omega} \rightarrow \mathbb{R}^n$ be a differentiable mapping and p be a regular value of f , where $p \notin f(\partial\Omega)$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain. Then the degree $\deg(f, \Omega, p)$ is defined by the formula

$$\deg(f, \Omega, p) := \sum_{y \in f^{-1}(p)} \text{sign}(\det J_f(y)).$$

The $\deg(f, \Omega, p)$ satisfies the following properties (cf. [33, 36]):

- (I) If $\deg(f, \overline{\Omega}, p) \neq 0$, then there exists an $x \in \Omega$ such that $f(x) = p$.
- (II) If D is a domain with $\overline{D} \subset \Omega$ and $p \in \mathbb{R}^n \setminus f(\partial D)$, then the degree $\deg(f, D, p)$ is a constant.

Let $D \subset \mathbb{R}^n$ be a domain and f be a real function from D into \mathbb{R}^n . If the Hölder coefficient

$$\|f\|_{\alpha, D} = \sup_{x, y \in D, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

is finite, then the function f is said to be (uniformly) Hölder continuous with exponent α in D , where $0 < \alpha \leq 1$. In this case, the Hölder coefficient serves as a seminorm. If the Hölder coefficient is merely bounded on compact subsets of D , then the function f is said to be locally Hölder continuous with exponent α in D . We denote by $C^\alpha(D, \mathbb{R}^n)$ the space consist of all locally Hölder continuous functions f from D into \mathbb{R}^n with exponent α (cf. [17, 20]).

Let \mathcal{PE}_f denote the class of functions u satisfying the Poisson equation $\Delta u = f$ with $J_u(0) - 1 = |u(0)| = 0$, where $u \in C^2(\mathbb{B}^n)$, i.e., twice continuously differentiable function in \mathbb{B}^n , and $f \in C^\alpha(\mathbb{B}^n, \mathbb{R}^n)$ with the constants $\alpha \in (0, 1)$ and $\|f\|_{\alpha, \mathbb{B}^n} < \infty$. We use \mathcal{PE}_f^M to denote the family of all functions u satisfying $u \in \mathcal{PE}_f$ with $|u(x)| \leq M$ for $x \in \mathbb{B}^n$, where M is a positive constant. Obviously, all bounded harmonic functions belong to \mathcal{PE}_f^M .

Theorem 6. *Let $u \in \mathcal{PE}_f^M$. Then there is a positive constant c_0 depending only on M , $\|f\|_{\alpha, \mathbb{B}^n}$ and n such that $\mathbb{B}^n(0, c_0) \subset u(\mathbb{B}^n)$.*

In fact, the bounded condition in Theorem 6 is necessary. The following example shows that there is no Landau-Bloch Theorem for functions $u \in \mathcal{PE}_f$ without the bounded condition.

Example 1.1. For $k \in \{1, 2, \dots\}$ and $x \in \mathbb{B}^n$, let $u_k(x) = (kx_1, x_2/k, x_3, \dots, x_n)$. Then u_k are harmonic and $J_{u_k}(0) - 1 = |u_k(0)| = 0$.

This example tells us that if $u : \mathbb{B}^n \rightarrow \mathbb{R}^n$ is a harmonic function on the unit ball with $J_u(0) - 1 = |u(0)| = 0$, then there is no an absolute constant $s > 0$ such that $\mathbb{B}^n(0, s)$ belongs to $u(\mathbb{B}^n)$. Thus the Theorem 6 does not hold for $u \in \mathcal{PE}_f$.

The proofs of Theorems 1, 2 and 3 will be given in Section 2. We will show Theorems 4 and 6 in the last part of this paper.

2. LIPSCHITZ TYPE SPACES ON HARMONIC FUNCTIONS

Proof of Theorem 1. We first prove the sufficiency. Since Ω is a Λ_ω -extension domain in \mathbb{R}^n , we see that for any $x, y \in \Omega$, by using (4), there is a rectifiable curve $\gamma \subset \Omega$ joining x to y such that

$$\begin{aligned} |f(x) - f(y)| &\leq \int_\gamma |\nabla f(\zeta)| ds(\zeta) \\ &\leq C \int_\gamma \frac{\omega(d_\Omega(\zeta))}{d_\Omega(\zeta)} ds(\zeta) \\ &\leq C\omega(|x - y|). \end{aligned}$$

Now we come to prove the necessity. Let $x = (x_1, \dots, x_n) \in \Omega$ and $r = d_\Omega(x)/2$. For all $y \in \mathbb{B}^n(x, r)$, using Poisson formula, we get

$$f(y) = \int_{\partial \mathbb{B}^n} P(y, \zeta) f(r\zeta + x) d\sigma(\zeta),$$

where $\zeta = (\zeta_1, \dots, \zeta_n) \in \partial \mathbb{B}^n$ and

$$P(y, \zeta) = \frac{r^2 - |y - x|^2}{|y - x - r\zeta|^2}.$$

By elementary calculations, for each $k \in \{1, 2, \dots, n\}$, we have

$$\frac{\partial P(y, \zeta)}{\partial y_k} = -2 \frac{[(y_k - x_k)|y - x - r\zeta|^2 + (r^2 - |y - x|^2)(y_k - r\zeta_k - x_k)]}{|y - x - r\zeta|^4}.$$

Then for all $y \in \mathbb{B}^n(x, r/2)$,

$$\begin{aligned} \left| \frac{\partial P(y, \zeta)}{\partial y_k} \right| &\leq 2 \frac{[|y_k - x_k||y - x - r\zeta|^2 + (r^2 - |y - x|^2)|y_k - r\zeta_k - x_k|]}{|y - x - r\zeta|^4} \\ &\leq 2 \frac{\left[\frac{r}{2} \left(\frac{3r}{2} \right)^2 + r^2 \left(\frac{3r}{2} \right) \right]}{\left(r - \frac{r}{2} \right)^4} = \frac{84}{r}, \end{aligned}$$

which implies that

$$\begin{aligned} |\nabla f(y)| &= \left[\sum_{k=1}^n f_{y_k}^2(y) \right]^{\frac{1}{2}} \\ &= \left\{ \sum_{k=1}^n \left(\left| \int_{\partial \mathbb{B}^n} \frac{\partial P(y, \zeta)}{\partial y_k} (f(r\zeta + x) - f(x)) d\sigma(\zeta) \right|^2 \right) \right\}^{\frac{1}{2}} \\ &\leq \sum_{k=1}^n \left| \int_{\partial \mathbb{B}^n} \frac{\partial P(y, \zeta)}{\partial y_k} (f(r\zeta + x) - f(x)) d\sigma(\zeta) \right| \\ &\leq \sum_{k=1}^n \int_{\partial \mathbb{B}^n} \left| \frac{\partial P(y, \zeta)}{\partial y_k} \right| |f(r\zeta + x) - f(x)| d\sigma(\zeta) \\ &\leq \sqrt{n} \int_{\partial \mathbb{B}^n} |\nabla P(y, \zeta)| |f(r\zeta + x) - f(x)| d\sigma(\zeta) \\ &\leq \frac{84n}{r} \int_{\partial \mathbb{B}^n} |f(r\zeta + x) - f(x)| d\sigma(\zeta) \\ &\leq \frac{84nC\omega(r)}{r} \\ &\leq 168nC \frac{\omega(d_\Omega(x))}{d_\Omega(x)}. \end{aligned}$$

If we take $y = x$, then we get the desired result. The proof of this theorem is complete. \square

Proof of Theorem 2. We first prove the necessity. For any $x, y \in \mathbb{B}^n$ with $x \neq y$, let $\varphi(t) = xt + (1-t)y$, where $t \in [0, 1]$. Since $|\varphi(t)| \leq t|x| + (1-t)|y|$, we see that

$$(5) \quad 1 - |\varphi(t)| \geq 1 - t|x| - |y| + t|y| \geq 1 - t + |y|(t-1) = (1-t)d(y)$$

and

$$(6) \quad 1 - |\varphi(t)| \geq 1 - t|x| - |y| + t|y| = 1 - t|x| - |y|(1-t) \geq 1 - t|x| - (1-t) = td(x).$$

By (5) and (6), we get

$$(1 - |\varphi(t)|)^2 \geq (1-t)td(x)d(y),$$

which implies

$$(7) \quad \frac{1}{1 - |\varphi(t)|} \leq \frac{1}{\sqrt{(1-t)td(x)d(y)}}.$$

For $t > 0$, by the monotonicity of $\omega(t)/t$, we know that

$$(8) \quad \omega(\lambda t) \leq \lambda \omega(t),$$

where $\lambda \geq 1$.

By (7) and (8), for any $x, y \in \mathbb{B}^n$ with $x \neq y$, we have

$$\begin{aligned} |f(x) - f(y)| &= \left| \int_0^1 \frac{df}{dt}(\varphi(t)) dt \right| \\ &\leq \sqrt{n}|x - y| \int_0^1 |\nabla f(\varphi(t))| dt \\ &\leq \sqrt{n}|x - y| \int_0^1 \frac{|\nabla f(\varphi(t))|}{\omega\left(\frac{1}{1-|\varphi(t)|}\right)} \omega\left(\frac{1}{1-|\varphi(t)|}\right) dt \\ &\leq C\sqrt{n}|x - y| \int_0^1 \omega\left(\frac{1}{1-|\varphi(t)|}\right) dt \\ &\leq C\sqrt{n}|x - y| \int_0^1 \omega\left(\frac{1}{\sqrt{(1-t)td(x)d(y)}}\right) dt \\ &\leq C\sqrt{n}|x - y| \omega\left(\frac{1}{\sqrt{d(x)d(y)}}\right) \int_0^1 \frac{1}{\sqrt{(1-t)t}} dt \\ &= C\sqrt{n}|x - y| \omega\left(\frac{1}{\sqrt{d(x)d(y)}}\right) \int_0^{\frac{\pi}{2}} \frac{2 \sin \theta \cos \theta}{\sqrt{\sin^2 \theta \cos^2 \theta}} d\theta \\ &= C\pi\sqrt{n}|x - y| \omega\left(\frac{1}{\sqrt{d(x)d(y)}}\right), \end{aligned}$$

which gives

$$\frac{|f(x) - f(y)|}{|x - y|} \leq \pi C \sqrt{n} \omega \left(\frac{1}{\sqrt{d(x)d(y)}} \right).$$

Now we prove the sufficiency part. For any $x, y \in \mathbb{B}^n$ with $x \neq y$, since

$$|\nabla f(x)| = \limsup_{y \rightarrow x} \frac{|f(y) - f(x)|}{|y - x|},$$

we see that

$$\limsup_{y \rightarrow x} \frac{|f(y) - f(x)|}{|y - x|} = |\nabla f(x)| \leq C \limsup_{y \rightarrow x} \omega \left(\frac{1}{\sqrt{d(x)d(y)}} \right) = C \omega \left(\frac{1}{d(x)} \right).$$

The proof of this theorem is complete. \square

Using arguments similar to those in the proof of [25, Lemma 2.5], we have the following lemma and so, we omit its proof.

Lemma 1. *Suppose that $f : \overline{\mathbb{B}^n}(a, r) \rightarrow \mathbb{R}$ is a continuous function in $\overline{\mathbb{B}^n}(a, r)$ and harmonic in $\mathbb{B}^n(a, r)$. Then*

$$|\nabla f(a)| \leq \frac{n\sqrt{n}}{r} \int_{\partial \mathbb{B}^n} |f(a + r\zeta) - f(a)| d\sigma(\zeta).$$

Proof of Theorem 3. (a) \iff (b) easily follows from Theorem 2. We only need to prove (a) \iff (c). We first prove (a) \implies (c). For any $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{B}^n$ and $t \in [0, 1]$, we have

$$d(x + t(y - x)) \geq d(x) - t|y - x|.$$

Suppose that $d(x) - t|y - x| > 0$. Then

$$\begin{aligned} |f(y) - f(x)| &= \left| \int_0^1 \frac{df}{dt}(\varsigma) dt \right| \\ &= \left| \sum_{k=1}^n (y_k - x_k) \int_0^1 \frac{df}{d\varsigma_k}(\varsigma) dt \right| \\ &\leq \left(\sum_{k=1}^n |y_k - x_k|^2 \right)^{\frac{1}{2}} \left[\sum_{k=1}^n \left(\int_0^1 \left| \frac{\partial f}{\partial \varsigma_k}(\varsigma) \right| dt \right)^2 \right]^{\frac{1}{2}} \\ &\leq \sqrt{n} |y - x| \int_0^1 |\nabla f(\varsigma)| dt \\ &\leq C \sqrt{n} |y - x| \int_0^1 \omega \left(\frac{1}{d(x) - t|y - x|} \right) dt \\ &= C \sqrt{n} \int_0^{|y-x|} \omega \left(\frac{1}{d(x) - t} \right) dt, \end{aligned}$$

which implies

$$\begin{aligned}
& \frac{1}{|\mathbb{B}^n(x, r)|} \int_{\mathbb{B}^n(x, r)} |f(\zeta) - f(x)| dV(\zeta) \\
& \leq \frac{C\sqrt{n}}{|\mathbb{B}^n(0, r)|} \int_{\mathbb{B}^n(0, r)} \left\{ \int_0^{|\xi|} \omega\left(\frac{1}{d(x) - t}\right) dt \right\} dV(\xi) \\
& = \frac{Cn\sqrt{n}}{r^n} \int_0^r \rho^{n-1} \left\{ \int_0^\rho \omega\left(\frac{1}{d(x) - t}\right) dt \right\} d\rho \\
& \leq \frac{Cn\sqrt{n}}{r^n} \int_0^r \left\{ \int_t^r \rho^{n-1} d\rho \right\} \omega\left(\frac{1}{r - t}\right) dt \\
& \leq \frac{C\sqrt{n}}{r^n} \int_0^r (r - t) (r^{n-1} + r^{n-2}t + \dots + t^{n-1}) \omega\left(\frac{1}{r - t}\right) dt \\
& \leq \frac{C\sqrt{n}}{r^n} r \omega\left(\frac{1}{r}\right) \int_0^r (r^{n-1} + r^{n-2}t + \dots + t^{n-1}) dt \\
& = C\sqrt{n} \left(\sum_{j=1}^n \frac{1}{j} \right) r \omega\left(\frac{1}{r}\right),
\end{aligned}$$

where $\varsigma = (\varsigma_1, \dots, \varsigma_n) = yt + (1 - t)x$.

Now we prove that (c) \implies (a). By Lemma 1, we have

$$|\nabla f(x)| \leq \frac{n\sqrt{n}}{\rho} \int_{\partial\mathbb{B}^n} |f(x + \rho\zeta) - f(x)| d\sigma(\zeta),$$

where $\rho \in (0, d(x)]$. Let $r = d(x)$. Then we have

$$\int_0^r |\nabla f(x)| \rho^n d\rho \leq \sqrt{n} \int_0^r \left(n\rho^{n-1} \int_{\partial\mathbb{B}^n} |f(x) - f(x + \rho\zeta)| d\sigma(\zeta) \right) d\rho,$$

which implies

$$\begin{aligned}
|\nabla f(x)| & \leq \frac{(n+1)\sqrt{n}}{2r^{n+1}} \int_0^r \left(n\rho^{n-1} \int_{\partial\mathbb{B}^n} |f(x) - f(x + \rho\zeta)| d\sigma(\zeta) \right) d\rho \\
& = \frac{(n+1)\sqrt{n}}{2r|\mathbb{B}^n(x, r)|} \int_{\mathbb{B}^n(x, r)} |f(\xi) - f(x)| dV(\xi) \\
& \leq \frac{(n+1)\sqrt{n}C}{2} \omega\left(\frac{1}{r}\right) \\
& = \frac{(n+1)\sqrt{n}C}{2} \omega\left(\frac{1}{d(x)}\right).
\end{aligned}$$

Therefore, (a) \iff (c). Since (a) \iff (b) and (a) \iff (c), we conclude that

$$(a) \iff (b) \iff (c).$$

The proof of the theorem is complete. \square

3. LANDAU-BLOCH THEOREM FOR FUNCTIONS IN $\mathcal{H}^p(\mathbb{B}^n, \mathbb{R}^n)$ AND \mathcal{PE}_f^M

The following lemmas are crucial for the proof of Theorem 4.

The following result is a Schwarz-Pick type lemma for harmonic functions in $\mathcal{H}(\mathbb{B}^n, \mathbb{R}^n)$.

Lemma 2. *Let $f \in \mathcal{H}(\mathbb{B}^n, \mathbb{R}^n)$ with $|f(x)| \leq M$ in \mathbb{B}^n , where M is a positive constant. Then*

$$(9) \quad \left| f(x) - \frac{1 - |x|}{(1 + |x|)^{n-1}} f(0) \right| \leq M \left[1 - \frac{1 - |x|}{(1 + |x|)^{n-1}} \right].$$

Proof. Without loss of generality, we assume that f is also harmonic on $\partial\mathbb{B}^n$. We first prove the inequality (9). By the Poisson integral formula, we have

$$(10) \quad f(x) = \int_{\partial\mathbb{B}^n} \frac{1 - |x|^2}{|x - \zeta|^n} f(\zeta) d\sigma(\zeta),$$

where $d\sigma$ denotes the normalized surface measure on $\partial\mathbb{B}^n$. By calculations, we have

$$\begin{aligned} \left| f(x) - \frac{1 - |x|}{(1 + |x|)^{n-1}} f(0) \right| &= \left| \int_{\partial\mathbb{B}^n} \left[\frac{1 - |x|^2}{|x - \zeta|^n} - \frac{1 - |x|}{(1 + |x|)^{n-1}} \right] f(\zeta) d\sigma(\zeta) \right| \\ &\leq \int_{\partial\mathbb{B}^n} \left[\frac{1 - |x|^2}{|x - \zeta|^n} - \frac{1 - |x|}{(1 + |x|)^{n-1}} \right] |f(\zeta)| d\sigma(\zeta) \\ &\leq M \int_{\partial\mathbb{B}^n} \left[\frac{1 - |x|^2}{|x - \zeta|^n} - \frac{1 - |x|}{(1 + |x|)^{n-1}} \right] d\sigma(\zeta) \\ &\leq M \left[1 - \frac{1 - |x|}{(1 + |x|)^{n-1}} \right] \end{aligned}$$

and the proof is complete. \square

A matrix-valued function $A(x) = (a_{i,j}(x))_{n \times n}$ is called *matrix-valued and real harmonic function* if each of its entries $a_{i,j}(x)$ is a real harmonic function from an open subset $\Omega \subset \mathbb{R}^n$ into \mathbb{R} .

Lemma 3. *Let $A(x) = (a_{i,j}(x))_{n \times n}$ be a matrix-valued harmonic mapping defined on the ball $\mathbb{B}^n(0, r)$. If $A(0) = 0$ and $|A(x)| \leq M$ in $\mathbb{B}^n(0, r)$, then*

$$|A(x)| \leq M \left[1 - \frac{r^{2n-2}(r - |x|)}{(r + |x|)^{2n-1}} \right].$$

Proof. For an arbitrary $\theta = (\theta_1, \dots, \theta_n)^T \in \partial\mathbb{B}^n$, we let

$$P_\theta(x) = A(x)\theta = (p_1(x), \dots, p_n(x)).$$

For every $\zeta \in \mathbb{B}^n$, let $F_\theta(\zeta) = P_\theta(r\zeta)$. By Lemma 2, we see that for all $\zeta \in \mathbb{B}^n$,

$$\left| F_\theta(\zeta) - \frac{1 - |\zeta|}{(1 + |\zeta|)^{n-1}} F_\theta(0) \right| \leq M \left[1 - \frac{1 - |\zeta|}{(1 + |\zeta|)^{n-1}} \right],$$

which gives

$$|P_\theta(x)| \leq M \left[1 - \frac{r^{n-2}(r - |x|)}{(r + |x|)^{n-1}} \right], \quad |x| < r.$$

The arbitrariness of θ yields the desired inequality. \square

Lemma 4. *Let $f \in \mathcal{H}(\mathbb{B}^n, \mathbb{R}^n)$ with $|f(x)| \leq M$ for $x \in \mathbb{B}^n$, where M is a positive constant. Then*

$$|f'(x)| \leq M \frac{2|x| + n(1 + |x|)}{1 - |x|^2}.$$

Proof. Let $f = (f_1, \dots, f_n)$ and $\theta = (\theta_1, \dots, \theta_n) \in \partial\mathbb{B}^n$. Without loss of generality, we assume that f is also harmonic on $\partial\mathbb{B}^n$. By the Poisson integral formula, we find that

$$f(x) = \int_{\partial\mathbb{B}^n} \frac{1 - |x|^2}{|x - \zeta|^n} f(\zeta) d\sigma(\zeta),$$

where $d\sigma$ denotes the normalized surface measure on $\partial\mathbb{B}^n$. Clearly,

$$(11) \quad \int_{\partial\mathbb{B}^n} \frac{d\sigma(\zeta)}{|x - \zeta|^n} = \frac{1}{1 - |x|^2}.$$

For each $j, k \in \{1, \dots, n\}$, we have

$$(f_j(x))_{x_k} = \int_{\partial\mathbb{B}^n} \frac{-2x_k|x - \zeta|^2 - n(1 - |x|^2)(x_k - \zeta_k)}{|x - \zeta|^{n+2}} f_j(\zeta) d\sigma(\zeta),$$

which gives

$$\begin{aligned} \left| \sum_{k=1}^n (f_j(x))_{x_k} \cdot \theta_k \right|^2 &= \left| \sum_{k=1}^n \int_{\partial\mathbb{B}^n} \frac{[2x_k|x - \zeta|^2 + n(1 - |x|^2)(x_k - \zeta_k)]\theta_k}{|x - \zeta|^{n+2}} f_j(\zeta) d\sigma(\zeta) \right|^2 \\ &= \left| \int_{\partial\mathbb{B}^n} \frac{\sum_{k=1}^n [2x_k|x - \zeta|^2 + n(1 - |x|^2)(x_k - \zeta_k)]\theta_k}{|x - \zeta|^{n+2}} f_j(\zeta) d\sigma(\zeta) \right|^2 \\ &\leq \left[\int_{\partial\mathbb{B}^n} \frac{[2|x||x - \zeta|^2 + n(1 - |x|^2)|x - \zeta|]|f_j(\zeta)|}{|x - \zeta|^{n+2}} d\sigma(\zeta) \right]^2 \\ &\leq \left[\int_{\partial\mathbb{B}^n} \frac{[2|x||x - \zeta| + n(1 - |x|^2)]^2}{|x - \zeta|^{n+2}} d\sigma(\zeta) \right] \\ &\quad \times \left[\int_{\partial\mathbb{B}^n} \frac{|f_j(\zeta)|^2}{|x - \zeta|^n} d\sigma(\zeta) \right] \end{aligned}$$

Then the relation (11) shows

$$\begin{aligned}
\sum_{j=1}^n \left| \sum_{k=1}^n (f_j(x))_{x_k} \cdot \theta_k \right|^2 &\leq \left[\int_{\partial \mathbb{B}^n} \frac{[2|x| |x - \zeta| + n(1 - |x|^2)]^2}{|x - \zeta|^{n+2}} d\sigma(\zeta) \right] \\
&\quad \times \left[\int_{\partial \mathbb{B}^n} \frac{\sum_{j=1}^n |f_j(\zeta)|^2}{|x - \zeta|^n} d\sigma(\zeta) \right] \\
&\leq \frac{M^2}{1 - |x|^2} \left[\int_{\partial \mathbb{B}^n} \frac{[2|x| |x - \zeta| + n(1 - |x|^2)]^2}{|x - \zeta|^{n+2}} d\sigma(\zeta) \right] \\
&\leq \frac{M^2}{1 - |x|^2} \left[\int_{\partial \mathbb{B}^n} \frac{[2|x| + n(1 + |x|)]^2}{|x - \zeta|^n} d\sigma(\zeta) \right] \\
&\leq \frac{M^2 [2|x| + n(1 + |x|)]^2}{(1 - |x|^2)^2}
\end{aligned}$$

whence

$$|f'(x)| \leq M \frac{2|x| + n(1 + |x|)}{1 - |x|^2}.$$

The proof of this lemma is complete. \square

Lemma E. ([22, Lemma 4]) *Let A be an $n \times n$ real (or complex) matrix with $|A| \neq 0$. Then for any unit vector $\theta \in \partial \mathbb{B}^n$, the inequality*

$$|A\theta| \geq \frac{|\det A|}{|A|^{n-1}}$$

holds.

Proof of Theorem 4. Without loss of generality, we assume that f is also harmonic on $\partial \mathbb{B}^n$, where $n \geq 3$. By the Poisson integral representation, we have

$$f(x) = \int_{\partial \mathbb{B}^n} \frac{1 - |x|^2}{|x - \zeta|^n} f(\zeta) d\sigma(\zeta)$$

in \mathbb{B}^n . By Jensen's inequalities, we obtain

$$|f(x)|^p \leq \int_{\partial \mathbb{B}^n} \frac{1 - |x|^2}{|x - \zeta|^n} |f(\zeta)|^p d\sigma(\zeta) \leq \frac{2\|f\|_p^p}{(1 - |x|)^{n-1}}$$

which gives

$$|f(x)| \leq \frac{2^{1/p} K_0}{(1 - |x|)^{(n-1)/p}},$$

where $K_0 = \|f\|_p$. For $\zeta \in \mathbb{B}^n$ and for a fixed $r \in (0, 1)$, let $F(\zeta) = f(r\zeta)/r$. Then

$$|F(\zeta)| \leq \frac{2^{1/p} K_0}{r(1 - r)^{(n-1)/p}} = K(r).$$

For each $\zeta \in \mathbb{B}^n(0, \sqrt{2}/2)$, using Lemma 4, we have

$$\begin{aligned} |F'(\zeta) - F'(0)| &\leq |F'(0)| + |F'(\zeta)| \\ &\leq nK(r) + \frac{K(r)[n + (n+2)|\zeta|]}{1 - |\zeta|^2} \\ &\leq K(r)[(3 + \sqrt{2})n + 2\sqrt{2}], \end{aligned}$$

which implies $F'(\zeta) - F'(0)$ is a bounded matrix-valued and real harmonic function in $\mathbb{B}^n(0, \sqrt{2}/2)$. By Lemma 3, for each $\zeta \in \mathbb{B}^n(0, \sqrt{2}/2)$, we have

$$\begin{aligned} |F'(\zeta) - F'(0)| &\leq M(r) \left[1 - \frac{\left(\frac{\sqrt{2}}{2}\right)^{n-2} (\frac{\sqrt{2}}{2} - |\zeta|)}{(\frac{\sqrt{2}}{2} + |\zeta|)^{n-1}} \right] \\ &\leq M(r) \cdot \frac{C_{n-1}^1 \left(\frac{\sqrt{2}}{2}\right)^{n-2} |\zeta| + \cdots + C_{n-1}^{n-1} |\zeta|^{n-1} + \left(\frac{\sqrt{2}}{2}\right)^{n-2} |\zeta|}{(\frac{\sqrt{2}}{2} + |\zeta|)^{n-1}} \\ &\leq M(r) |\zeta| \frac{\left[(1 + \frac{\sqrt{2}}{2})^{n-1} + \left(\frac{\sqrt{2}}{2}\right)^{n-2} - \left(\frac{\sqrt{2}}{2}\right)^{n-1} \right]}{(\frac{\sqrt{2}}{2} + |\zeta|)^{n-1}} \\ &\leq M(r) \left[(1 + \sqrt{2})^{n-1} + \sqrt{2} - 1 \right] |\zeta|, \end{aligned}$$

where $M(r) = K(r)[(3 + \sqrt{2})n + 2\sqrt{2}]$ and $C_n^k = \binom{n}{k}$ ($k = 1, 2, \dots, n$) denote the binomial coefficients.

Since for each $\theta \in \partial\mathbb{B}^n$, Lemmas 4 and E imply

$$|F'(0)\theta| \geq \frac{J_F(0)}{|F'(0)|^{n-1}} \geq \frac{1}{[nK(r)]^{n-1}}.$$

Let ζ' and ζ'' be two distinct points in $\mathbb{B}^n(0, \rho(r))$ with

$$\rho(r) = \frac{1}{[nK(r)]^{n-1} M(r) [(1 + \sqrt{2})^{n-1} + \sqrt{2} - 1]},$$

and let $[\zeta', \zeta'']$ denote the segment connecting ζ' and ζ'' . Set

$$d\zeta = \begin{pmatrix} d\zeta_1 \\ \vdots \\ d\zeta_n \end{pmatrix}.$$

Then we have

$$\begin{aligned} |F(\zeta') - F(\zeta'')| &\geq \left| \int_{[\zeta', \zeta'']} F'(0) d\zeta \right| - \left| \int_{[\zeta', \zeta'']} (F'(\zeta) - F'(0)) d\zeta \right| \\ &> |\zeta' - \zeta''| \left\{ \frac{1}{[nK(r)]^{n-1}} - M(r) [(1 + \sqrt{2})^{n-1} + \sqrt{2} - 1] \rho(r) \right\} \\ &= 0. \end{aligned}$$

This observation shows that F is univalent in $\mathbb{B}^n(0, \rho(r))$. Furthermore, for any ζ_0 with $|\zeta_0| = \rho(r)$, we have

$$\begin{aligned} |F(\zeta_0) - F(0)| &\geq \left| \int_{[0, \zeta_0]} F'(0) d\zeta \right| - \left| \int_{[0, \zeta_0]} (F'(\zeta) - F'(0)) d\zeta \right| \\ &\geq \rho(r) \left\{ \frac{1}{[nK(r)]^{n-1}} - M(r) \left[(1 + \sqrt{2})^{n-1} + \sqrt{2} - 1 \right] \rho(r)/2 \right\} \\ &= \frac{\rho(r)}{2[nK(r)]^{n-1}} \\ &> 0. \end{aligned}$$

Therefore, $f(\mathbb{B}^n)$ contains a univalent ball $\mathbb{B}^n(0, R)$, where

$$\begin{aligned} R &\geq \max_{0 < r < 1} \left\{ \frac{\rho(r)}{2[nK(r)]^{n-1}} \right\} \\ &= \max_{0 < r < 1} \left\{ \frac{1}{2[nK(r)]^{2n-2} M(r) [(1 + \sqrt{2})^{n-1} + \sqrt{2} - 1]} \right\}. \end{aligned}$$

The theorem is proved. \square

Proof of Theorem 6. If we suppose that this result is not true, then there is a sequence $\{a_k\}$ and a sequence of functions $\{u_k\}$ with $u_k \in \mathcal{PE}_f^M$, such that $\{a_k\}$ tends to 0 and $a_k \notin u_k(\mathbb{B}^n)$, where $a_k > 0$ for $k \in \{1, 2, \dots\}$. By [17, Theorem 4.6 and Corollary 4.7], we know that there is a subsequence $\{g_k\}$ of $\{u_k\}$ which converges uniformly on compact subsets of \mathbb{B}^n to a function g . Note that for each k , the function $h_k = g_k - g_1$ is harmonic. Hence the sequence $\{h_k\}$ converges uniformly on compact subsets of \mathbb{B}^n to $g - g_1$ and therefore, the partial derivatives of g_k converge uniformly on compact subsets of \mathbb{B}^n to the partial derivatives of g . In particular, $g_k(0) \rightarrow g(0)$ and $J_{g_k}(0) \rightarrow J_g(0)$, and therefore, $g \in \mathcal{PE}_f^M$. Since $J_g(0) - 1 = |g(0)| = 0$, there are $0 < r_0 < 1$ and $c_1 > 0$ such that $J_g > 0$ on $\overline{\mathbb{B}^n(0, r_0)}$, $g(\mathbb{B}^n(0, r_0)) \supset \overline{\mathbb{B}^n(0, c_1)}$ and $|g(x)| \geq c_1$ for $x \in \partial\mathbb{B}^n(0, r_0)$.

Set $c_2 = c_1/2$, $B_{r_0} = \mathbb{B}^n(0, r_0)$ and $\overline{B_{c_2}} = \overline{\mathbb{B}^n(0, c_2)}$. Then there is a k_0 such that $|g_k(x)| \geq c_2$ for $k \geq k_0$ and $J_{g_k} > 0$ on $\overline{B_{r_0}}$. Since $\deg(g_k, B_{r_0}, 0) \geq 1$, by the degree property (II) in page 5, we see that $\deg(g_k, B_{r_0}, y) \geq 1$ for $y \in B_{c_2}$ and $k \geq k_0$. Hence $g_k(B_{r_0}) \supset B_{c_2}$ for $k \geq k_0$ and this leads a contradiction. The proof of the theorem is complete. \square

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SH. CHEN, DEPARTMENT OF MATHEMATICS AND COMPUTATIONAL SCIENCE, HENGYANG NORMAL UNIVERSITY, HENGYANG, HUNAN 421008, PEOPLE'S REPUBLIC OF CHINA.

E-mail address: mathechen@126.com

M. MATELJEVIĆ, UNIVERSITY OF BELGRADE, FACULTY OF MATHEMATICS STUDENTSKI TRG 16, 11000 BELGRADE, SERBIA.

E-mail address: miodrag@matf.bg.ac.rs

S. PONNUSAMY, INDIAN STATISTICAL INSTITUTE (ISI), CHENNAI CENTRE, SETS (SOCIETY FOR ELECTRONIC TRANSACTIONS AND SECURITY), MGR KNOWLEDGE CITY, CIT CAMPUS, TARAMANI, CHENNAI 600 113, INDIA.

E-mail address: samy@isichennai.res.in, samy@iitm.ac.in

X. WANG, DEPARTMENT OF MATHEMATICS, HUNAN NORMAL UNIVERSITY, CHANGSHA, HUNAN 410081, PEOPLE'S REPUBLIC OF CHINA.

E-mail address: mathshida@gmail.com